Journal of Statistical Physics, Vol. 26, No. 3, 1981

Infinite Clusters in Percolation Models

C. M. Newman[1] **and L. S. Schulman**[2]

Received June 19, 1980

The qualitative nature of infinite clusters in percolation models is investigated. The results, which apply to both independent and correlated percolation in any dimension, concern the number and density of infinite clusters, the size of their external surface, the value of their (total) surface-to-volume ratio, and the fluctuations in their density. In particular it is shown that N_0, the number of distinct infinite clusters, is either 0, 1, or ∞ and the case $N_0 = \infty$ (which might occur in sufficiently high dimension) is analyzed.

KEY WORDS: Percolation; infinite clusters; cluster density; surface-to-volume ratio.

1. INTRODUCTION AND RESULTS

In this paper, we present results concerning the number and nature of infinite clusters which are applicable to a large class of percolation models. Our analysis is based primarily on ergodic theory and measure theory. In order to give a precise statement of the results and a complete presentation of the proofs, we use a rather formal mathematical style in this paper. For an alternate description (without proofs) from a more physical point of view, see Ref. 1. In this section, we describe the class of models considered, give some basic definitions, and state our results; the proofs are presented in Section 2. We only treat properties of the infinite clusters; rigorous results concerning properties of large finite clusters are discussed in Refs. 2 and 3.

For ease of exposition, we restrict attention in this paper to site percolation in the simple d-dimensional cubic lattice with nearest-neighbor

[1] Mathematics Department, University of Arizona, Tucson, Arizona 85721, U.S.A. Alfred P. Sloan Research Fellow, Research supported in part by National Science Foundation grant No. MCS 77-20683 and by the U.S.–Israel Binational Science Foundation.

[2] Department of Physics, Technion, Haifa, Israel. Research supported in part by the U.S.–Israel Binational Science Foundation.

0022-4715/81/1100-0613$03.00/0

bonds; our results, however, apply (with simple modifications) to rather general lattices and to bond percolation. The models we consider may be defined by a lattice of (site occupation) random variables, $\{X_k : k \in \mathbb{Z}^d\}$, where each X_k either takes the value 1 (corresponding to site k occupied) or the value 0 (corresponding to site k not occupied). Such a model may equivalently be defined by the joint distribution P of $\{X_k\}$ which is a probability measure on the configuration space,

$$\Omega = \{0,1\}^{\mathbb{Z}^d} = \left\{\omega = \left(\omega_k : k \in \mathbb{Z}^d\right): \qquad \text{each } \omega_k = 0 \text{ or } 1\right\}$$

(with the standard definition of measurable sets); we assume (without loss of generality) that (Ω, P) is the underlying probability space with $X_k(\omega) = \omega_k$.

Example 1. In classical independent percolation of parameter $p \in [0, 1]$, the X_k's are independent identically distributed random variables and P is the infinite product measure:

$$P = \prod_{k \in \mathbb{Z}^d} \left[p\delta(\omega_k - 1) + (1-p)\delta(\omega_k) \right] \tag{1}$$

Example 2. In correlated percolation, the X_k's are dependent random variables. A class of examples can be obtained by starting with an infinite volume spin-1/2 (i.e., ± 1 valued) Ising model $\{\sigma_k : k \in \mathbb{Z}^d\}$ and defining $X_k = (1 + \sigma_k)/2$. For example, with inverse temperature β and Hamiltonian

$$H(\{\sigma_k\}) = -\sum_j h\sigma_j - \sum_j \sum_k J(k-j)\sigma_j\sigma_k$$

P is formally proportional to

$$\exp\left[-\beta H(\{2\omega_k - 1\})\right] \prod_k \left[\delta(\omega_k - 1) + \delta(\omega_k)\right]$$

The X_k's will be dependent if $J \not\equiv 0$ and $\beta \in (0, \infty)$.

For any $j \in \mathbb{Z}^d$ we consider the shift operator T_j which acts either on configurations $\omega \in \Omega$, events (i.e., measurable sets) $W \subset \Omega$, measures P on Ω, or random variables X on Ω according to

$$(T_j\omega)_k = \omega_{k-j}, \qquad T_j W = \{T_j\omega : \omega \in W\}$$

$$(T_j P)(W) = P(T_{-j}W), \qquad (T_j X)(\omega) = X(T_{-j}\omega)$$

For each $k \in \mathbb{Z}^d$ and $\eta = 0$ or 1 we consider the measures P_k^η on $\Omega_k = \{0,1\}^{\mathbb{Z}^d \setminus \{k\}}$, defined so that for $U \subset \Omega_k$,

$$P_k^\eta(U) = P(U \times \{\omega_k = \eta\}) \tag{2}$$

P_k is the conditional distribution of $\{X_j : j \neq k\}$ conditioned on $X_k = \eta$.

Throughout this paper, we assume the following three hypotheses on P (or equivalently on $\{X_k\}$):

Hypothesis A. P is translation invariant; i.e., for any $j \in \mathbb{Z}^d$, $T_j P = P$.

Hypothesis B. P is translation ergodic; i.e., if $j \neq 0$ and W is an event such that $T_j W = W$, then $P(W) = 0$ or 1.

Hypothesis C. For any k, P_k^0 and P_k^1 are equivalent measures; i.e., if $U \subset \Omega_k$, $\eta = 0$ or 1, and $P_k^\eta(U) \neq 0$, then $P_k^{1-\eta}(U) \neq 0$.

In the independent percolation of Example 1, Hypothesis A is immediate, Hypothesis B is known to be valid (see, e.g., Ref. 4, Theorem 1.2), and Hypothesis C follows (for $p \neq 0, 1$) from the simple fact that

$$p^{-1} P_k^1 = (1-p)^{-1} P_k^0 = \prod_{j \neq k} \left[p\delta(\omega_j - 1) + (1-p)\delta(\omega_j) \right] \qquad (3)$$

In the Ising model percolation of Example 2, Hypotheses A and B will be valid if $\{\sigma_k\}$ represents a translation-invariant pure phase. Hypothesis C (for $\beta \neq 0, \infty$) is equivalent to assuming that the energy shift due to a single spin flip is finite since

$$dP_k^1 / dP_k^0 = \exp\left[-\beta\left(H\{\sigma_k = 1; \sigma_j, j \neq k\} - H\{\sigma_k = 0; \sigma_j, j \neq k\}\right)\right] \quad (4)$$

where the left-hand side of (4) denotes the Radon–Nikodym derivative, so that we have $0 < dP_k^1/dP_k^0 < \infty$ (which yields Hypothesis C) if

$$\beta \left| 2h + 2 \sum_{j \neq k} \left[J(k-j) + J(j-k) \right] \sigma_j \right| < \infty$$

which is the case if $\sum_j |J(j)| < \infty$. Hypothesis C is equivalent to the following reformulation, which can be more convenient in certain situations:

Hypothesis C′. For any k,

$$0 < E(X_k \mid \{X_j : j \neq k\}) < 1 \qquad \text{(w.p.o.)}$$

where w.p.o. means "with probability one."

Given a particular configuration $\omega \in \Omega$, we say that i is connected to j if i and j are both occupied and there is a nearest-neighbor path of occupied sites leading from i to j; i.e., if there is some finite sequence $i_0, i_1, \ldots, i_n \in \mathbb{Z}^d$ with $i_0 = i$, $i_n = j$, $\|i_j - i_{j-1}\| = 1$ for $j = 1, \ldots, n$ (where $\|\cdot\|$ denotes Euclidean distance) and $X_{i_j} = 1$ for $j = 0, 1, \ldots, n$. We define $C(j)$, the cluster belonging to j, as $C(j) = \{i : i \text{ is connected to } j\}$; note that

$C(j)=\varnothing$ if j is not occupied. A set $C\subset\mathbb{Z}^d$ is called a cluster if $C=C(j)$ for some j and is called an infinite cluster if in addition $|C|=\infty$ where $|\cdot|$ denotes cardinality. The percolation probability is $\rho=P(|C(k)|=\infty)$ (which is independent of k). We define for any $F\subset\mathbb{Z}^d$ its lower density, $\underline{D}(F)$ and upper density $\overline{D}(F)$ as

$$\underline{D}(F)=\liminf_{n\to\infty}\frac{|F\cap R_n|}{n^d},\qquad \overline{D}(F)=\limsup_{n\to\infty}\frac{|F\cap R_n|}{n^d}$$

where

$$R_n=\left\{j=(j_1,\ldots,j_d)\in\mathbb{Z}^d: -n/2\leqslant j_l<n/2,\qquad l=1,\ldots,d\right\}\tag{5}$$

F is said to have a density $D(F)$ if $\underline{D}(F)=\overline{D}(F)=D(F)$. We denote by H the set of all clusters C and define

$$\begin{aligned}&H_0=\{C\in H:|C|=\infty\},\qquad N_0=|H_0|\\&F_0=\left\{j\in\mathbb{Z}^d: j\in C\qquad\text{for some}\quad C\in H_0\right\}\end{aligned}\tag{6}$$

H_0 is the set of infinite clusters, N_0 the number of infinite clusters, and F_0 the union of infinite clusters.

N_0 is a random variable, and for each $k=0,1,2,\ldots$ or ∞ there are of course many configurations ω with $N_0(\omega)=k$. The following theorem shows, independently of the particular model, that except for $k=0,1,\infty$, these configurations have probability zero of occurring; in particular one has either (depending on d and the parameters of the model) no infinite clusters (w.p.o.), or a single infinite cluster (w.p.o.), or infinitely many distinct infinite clusters (w.p.o.). (Note that the theorem refers to clusters of occupied sites only; a single infinite cluster of occupied sites may coexist with a single infinite cluster of unoccupied sites.) The impossibility of $N_0>1$ for $d=2$ was shown in Refs. 5 and 6 and there have appeared heuristic arguments (see, for example, Ref. 7) which attempt to extend this result to general d. It seems, however, that the possibility that $N_0=\infty$ or that zero-density infinite clusters might exist (see Theorem 2 below) has not been previously discussed in the literature. The following theorem also shows that there are no infinite clusters if $\rho=0$ (which is believed to be the case when $p=p_c\equiv\inf\{p:\rho>0\}$ in independent percolation).

Theorem 1. Exactly one of the following three events has probability 1: (i) $N_0=0$; (ii) $N_0=1$; (iii) $N_0=\infty$. If $N_0\neq 0$ (w.p.o.), then $\rho>0$ and $D(F_0)=\rho$ (w.p.o.).

The next theorem further analyzes the $N_0=\infty$ case into several disjoint possibilities. It shows that the appearance of zero-density (or at least zero-lower-density) clusters is a necessity. We define the following:

$$\begin{aligned} H_1 &= \{C \in H_0 : D(C) \text{ exists and } D(C) > 0\} \\ H_2 &= \{C \in H_0 : D(C) = 0\}, \\ H_3 &= \{C \in H_0 : \underline{D}(C) = 0 \quad \text{and} \quad \overline{D}(C) > 0\} \end{aligned} \tag{7}$$

Clusters in the class H_1 will be called dense; in the class H_2, filamentary; in the class H_3, rough. For $l = 1,2,3$, we define, in analogy with (6),

$$N_l = |H_l|, \qquad F_l = \{j \in \mathbb{Z}^d : j \in C \quad \text{for some} \quad C \in H_l\}$$

Theorem 2. Assume $N_0 = \infty$ (w.p.o). It follows that $H_0 = H_1 \cup H_2 \cup H_3$ (w.p.o.) and $N_0 = N_1 + N_2 + N_3$ (w.p.o.); in particular exactly one of the following four events has probability 1: (i) $N_2 = \infty$ and $N_1 = N_3 = 0$; (ii) $N_2 = \infty$, $N_1 = 1$, and $N_3 = 0$; (iii) $N_3 = \infty$ and $N_1 = N_2 = 0$; (iv) $N_2 = N_3 = \infty$ and $N_1 = 0$. For each $l = 1,2,3$ there is a constant $\rho_l > 0$ such that $D(F_l) = \rho_l$ (w.p.o.); $\rho_1 + \rho_2 + \rho_3 = \rho$ and if $N_l \neq 0$ (w.p.o.), then $\rho_l > 0$.

Remark. We suspect that cases (iii) and (iv) of Theorem 2 do not occur (at least not in physically interesting models) but that cases (i) and (ii) do occur in sufficiently high dimension. If this happened in independent percolation, then the usual critical point, $p = p_c \equiv \inf\{p : \rho > 0\}$, should correspond to the transition from $N_0 = 0$ to case (i) of Theorem 2, while other critical points should exist corresponding first to the transition from case (i) to case (ii) and then from case (ii) to $N_0 = 1$. Our next theorem suggests some of the phenomena associated with zero-density infinite clusters. It would be of considerable interest to obtain either a rigorous proof or numerical evidence for the existence of filamentary clusters.

The distance from $j \in \mathbb{Z}^d$ to $F \subset \mathbb{Z}^d$ is defined in the usual way as

$$\gamma(j, F) = \inf\{\|k - j\| : k \in F\}$$

The set of "close-encounter sites" between (distinct) infinite clusters of types l and m ($l, m = 0,1,2,3$) is defined as

$$\begin{aligned} A_{lm} = \{j \in \mathbb{Z}^d: \quad &\exists C_1 \in H_l, C_2 \in H_m \quad \text{with} \quad C_1 \neq C_2 \\ &\text{such that} \quad \gamma(j, C_1) = \gamma(j, C_2) = 1\} \end{aligned} \tag{8}$$

These are vacant sites, which, if occupied, would connect two (or more) previously distinct infinite clusters (of specified type). To formally define the dual notion of "cutting sites" which belong to an infinite cluster and, if vacated, would disconnect that cluster into two (or more) distinct infinite clusters of specified type, we first define for $j \in \mathbb{Z}^d$, $\omega \in \Omega$, the "j-vacated" configuration, $\omega[j]$, as

$$(\omega[j])_k = \begin{cases} \omega_k, & k \neq j \\ 0, & k = j \end{cases}$$

We then define, for $l = 0, 1, 2, 3$, $H_l[j]$ as

$$(H_l[j])(\omega) = H_l(\omega[j]) \tag{9}$$

and the set of cutting sites, for $l, m = 0, 1, 2, 3$,

$$\begin{aligned} B_{lm} = \{ j \in \mathbb{Z}^d \colon \quad \exists C_1 \in H_l[j], C_2 \in H_m[j] \quad \text{with} \\ C_1 \neq C_2, \quad \text{such that} \quad (C_1 \cup C_2) \subset C(j) \} \end{aligned} \tag{10}$$

The next theorem shows that whenever distinct infinite clusters of type l and m occur, then there is a positive density of both close-encounter and cutting sites. In the case of independent percolation, these two densities are related to each other in a simple way (see Theorem 5 below).

Theorem 3. Choose $l, m \in \{0, 1, 2, 3\}$. $A_{lm} = \varnothing = B_{lm}$ unless

$$N_l \neq 0, \qquad N_m \neq 0, \qquad \text{and} \qquad N_l + N_m = \infty \qquad \text{(w.p.o.)} \tag{11}$$

If (11) is valid, then there are constant $\alpha_{lm} > 0$, $\beta_{lm} > 0$ such that $D(A_{lm}) = \alpha_{lm}$ (w.p.o.) and $D(B_{lm}) = \beta_{lm}$ (w.p.o.).

We define the external surface $\partial^e F$ of a set $F \subset \mathbb{Z}^d$ as $\partial^e F = \{ j \in F^c :$ $\gamma(j, F)\} = 1$ and there exists an infinite sequence of distinct points j_0, $j_1, \ldots \in F^c$ with $j_0 = j$ and $\|j_{m+1} - j_m\| = 1$ for all $m\}$; i.e., the external surface of F consists of nearest neighbors of F which are "connected to ∞" through points outside of F. The following theorem implies that clusters of positive density have either no external surface or else a large one in the sense that the asymptotic surface-to-volume ratio is bounded away from zero. There are other possible definitions of external surface, but the same result would apply to most of them. For a review of numerical results concerning the external surface of large finite clusters, see Ref. 8, Sec. 4.1.2.

Theorem 4. For $l = 0, 1, 2, 3$, if $P(\partial^e F_l \neq \varnothing) > 0$, then there is a constant $\nu_l > 0$ such that $D(\partial^e F_l) = \nu_l$ (w.p.o.).

Remark. Given $\omega \in \Omega$, let ω^* be defined by $(\omega^*)_k = 1 - \omega_k$. It is easily verified that if $F_l(\omega) \neq \varnothing$ (w.p.o.) and $F_0(\omega^*) \neq \varnothing$ (w.p.o.) then $\partial^e(F_l(\omega)) \neq \varnothing$ (w.p.o.). It follows that in independent percolation, if for a given d, $p_c \equiv \inf\{p : \rho > 0\} < 1/2$, then there is a nonempty external surface to F_0 (w.p.o.) for $p_c < p < 1 - p_c$. It can also be shown in independent percolation that if there exists an infinite cluster of positive density (w.p.o.)

for a certain p and dimension d, then there exists an infinite positive-density cluster for the same p and dimension $d+1$; since for $d=2$ and $p > p_c(2)$ there is a single infinite cluster (Refs. 5 and 6), it follows that if d is sufficiently large so that $p_c(d) < 1 - p_c(2)$, then there will be an interval $p_c(2) < p < 1 - p_c(d)$ in which there is an infinite positive-density cluster with a nonempty external surface.

One may also define the (total) surface ∂F of $F \subset \mathbb{Z}^d$ as

$$\partial F = \{ j \in F^c : \gamma(j, F) = 1 \}$$

It follows from Hypothesis C that ∂F_l is nonempty (w.p.o.) if F_l is nonempty (w.p.o.) and an analog of Theorem 4 applies to give a well-defined density to ∂F_l. What is more surprising, perhaps, is that in the case of independent percolation (see Example 1 above), the asymptotic surface-to-volume ratio $D(\partial F_l)/D(F_l)$ has (at least for $l = 0, 1$) the explicit value $(1-p)/p$. This value was first obtained by nonrigorous arguments in Ref. 9 and independently by numerical methods in Ref. 10. A rigorous proof was recently obtained (for each $C \in H_0$) by Klein and Shamir;[11] their result motivated us to search for a simpler proof (based on a kind of duality), which led to the next theorem. In it, we generalize the result to an analogous result about the ratio of the density of close encounter sites to that of cutting sites [see equations (8) and (10) for the appropriate definitions]. We also find that in the case of dependent percolation models satisfying the inequalities of Fortuin, Kastelyn and Ginibre (FKG)[12] (see below), the equation $D(\partial F_l)/D(F_l) = P(X_0 = 0)/P(X_0 = 1)$ is replaced by an inequality, which, in the case of the Ising models of Example 2, bounds the surface-to-volume ratio in terms of the magnetization. A brief discussion of numerical results concerning the surface-to-volume ratio in Ising models is given in Ref. 10. Numerical results concerning the critical point in independent percolation are given in Refs. 13 and 14.

A finite family of random variables $Y_1, \ldots, Y_m$ is said to satisfy the FKG inequalities, if for any real functions f_1, f_2 on $\mathbb{R}^m$ which are coordinate-wise increasing and such that $\tilde{f}_j \equiv f_j(Y_1, \ldots, Y_m)$ has $E(\tilde{f}_j^2) < \infty$ $(j = 1, 2)$, it follows that

$$\mathrm{Cov}(\tilde{f}_1, \tilde{f}_2) \equiv E(\tilde{f}_1 \tilde{f}_2) - E(\tilde{f}_1)E(\tilde{f}_2) \geqslant 0$$

An infinite family is said to satisfy the FKG inequalities if every finite subfamily does. The percolation model of Example 1 satisfies the FKG inequalities[5] as does that of Example 2 providing $J(k) \geqslant 0\ \forall k$.[12]

Theorem 5. In the case of independent percolation, let $(F, F^*) = (F_0, \partial F_0)$ or $(F_1, \partial F_1)$ or (B_{lm}, A_{lm}) (for $l, m = 0, 1, 2, 3$), and assume $F \neq \varnothing$ (w.p.o.); then $D(F) > 0$, $D(F^*) > 0$, and

$$D(F^*)/D(F) = (1-p)/p = P(X_0 = 0)/P(X_0 = 1) \tag{12}$$

If $\{X_k : k \in \mathbb{Z}^d\}$ satisfies the FKG inequalities, then for $(F, F^*) = (F_0, \partial F_0)$ or $(F_1, \partial F_1)$ we have

$$D(F^*)/D(F) \leqslant P(X_0 = 0)/P(X_0 = 1) \tag{13}$$

For each $i \in \mathbb{Z}^d$, define the cube R_n^i of side length n centered near ni as

$$R_n^i = \{ j \in \mathbb{Z}^d : j = ni + k \quad \text{for some} \quad k \in R_n \} = R_n + ni$$

where R_n is defined in (5), and define

$$Z_{l,n}^i = |F_l \cap R_n^i| - E(|F_l \cap R_n^i|) = |F_l \cap R_n^i| - \rho_l n^d \tag{14}$$

The fact that $D(F_l) = \rho_l$ is equivalent to

$$\forall i, \qquad Z_{l,n}^i / n^d \to 0 \qquad \text{(w.p.o.)} \tag{15}$$

The next theorem generalizes this by considering infinite cluster density fluctuations and states that under certain conditions these are of strength $n^{-d/2}$ and normally distributed as are the density fluctuations in a gas or liquid (away from the critical point). Note that the result *only applies for* $l = 0$ *or* 1 and implies that for $l = 0, 1$, the random variables $\{ \mu_l^i : i \in \mathbb{Z}^d\}$ defined by

$$\mu_l^i = \begin{cases} 1 & \text{if} \quad i \in F_l \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

behave on a macroscopic scale as though they were independent with mean ρ_l and some positive variance V_l.

Theorem 6. Assume that $\{X_k : k \in \mathbb{Z}^d\}$ satisfies the FKG inequalities. For $l = 0$ or 1, suppose $H_l \neq \varnothing$ (w.p.o.), and define $K_l(i)$ on $\mathbb{Z}^d$ by

$$K_l(i - j) = \mathrm{Cov}(\mu_l^i, \mu_l^j) = P(i, j \quad \text{both belong to } H_l) - \rho_l^2 \tag{17}$$

$K_l(i) \geqslant 0$ and $K_l(0) = \rho_l - \rho_l^2 > 0$. If

$$V_l \equiv \sum_{i \in \mathbb{Z}^d} K_l(i) < \infty \tag{18}$$

then $\{\tilde{Z}_{l,n}^i \equiv Z_{l,n}^i / V_l^{1/2} n^{d/2} : i \in \mathbb{Z}^d\}$ tend to independent standard normal random variables as $n \to \infty$ in the sense that for any N and distinct $i_1, \ldots, i_N \in \mathbb{Z}^d$ and any $a_1 \leqslant b_1, \ldots, a_N \leqslant b_N$,

$$P(a_1 \leqslant \tilde{Z}_{l,n}^{i_1} \leqslant b_1, \ldots, a_N \leqslant \tilde{Z}_{l,n}^{i_N} \leqslant b_N) \to \prod_{j=1}^{N} \left[(2\pi)^{-1/2} \int_{a_j}^{b_j} \exp(-z^2/2)\, dz \right] \tag{19}$$

Remark. An analog of Theorem 6 applies to the density fluctuations of $F_l \cup \partial F_l$ ($l = 0$ or 1); this result when combined with Theorem 6 itself gives some information on the fluctuations in surface-to-volume ratio of F_0

and F_1. For a discussion of such fluctuations in large finite clusters, see Ref. 8, Sec. 4, and for results on infinite clusters, see Ref. 11.

Our final result concerns how the various cases of Theorems 1 and 2 are related to the asymptotic behavior as $\|i\| \to \infty$ of the quantity $G(i)$ defined as

$$G(i-j) = P(i \text{ is connected to } j) \tag{20}$$

Note that i is not connected to j unless i and j are both occupied. We also define

$$Q_n = n^{-d} \sum_{j \in R_n} G(j) \tag{21}$$

Theorem 7. If

$$\liminf_{n \to \infty} \sum_{j \in \partial R_n} G(j) = 0 \tag{22}$$

then $H_0 = \varnothing$ (w.p.o.); if

$$\sum_{j \in \mathbb{Z}^d} \|j\|^{-(d-1)} G(j) < \infty \tag{23}$$

then $H_0 = H_2$ (w.p.o.); if

$$\liminf_{n \to \infty} Q_n = 0 \tag{24}$$

then $H_1 = \varnothing$ (w.p.o.). If $H_0 = H_2$ (w.p.o.), then

$$\lim_{n \to \infty} Q_n = 0 \tag{25}$$

If $H_1 \neq \varnothing$ (w.p.o.), then

$$\lim_{n \to \infty} Q_n = (\rho_1)^2 > 0 \tag{26}$$

Remark. One can use $G(j)$ to define a "mean effective dimension,"

$$\delta = \sup\left\{ \delta' : \lim_{n \to \infty} n^{-\delta'} \sum_{j \in R_n} G(j) = \infty \right\} \tag{27}$$

If one defines for a particular infinite cluster C its effective dimension

$$\underline{\delta}(C) = \sup\left\{ \delta' : \lim_{n \to \infty} n^{-\delta'} |C \cap R_n| = \infty \right\} \tag{28}$$

then it can be shown that

$$\forall C \in H_0, \underline{\delta}(C) \leqslant \delta \qquad \text{(w.p.o.)} \tag{29}$$

It would be interesting to obtain some nontrivial estimates on δ and $\underline{\delta}(C)$. For more discussion of effective and fractal dimension in percolation and elsewhere, see Refs. 14, 15, and 16.

2. PROOFS

The proofs of Theorems 1–5 are based on the following two propositions together with some geometric arguments. The first proposition, which follows from Hypotheses A and B, is a standard fact of ergodic theory.[17,18]

Proposition 8. Suppose $X \in L^1(\Omega, P)$; then

$$\lim_{n\to\infty} n^{-d} \sum_{j\in R_n} T_j X = E(X) \qquad \text{(w.p.o.)} \tag{30}$$

The next proposition follows from Hypothesis C and is the key to most of our results; it says that by altering configurations at a fixed finite number of sites, one cannot reduce the probability of an event from a positive value to zero. Its validity in the independent percolation of Example 1 (for $p \neq 0, 1$) follows from the facts [see Eq. (3)] that each alteration at a single site introduces a factor of $[(1-p)/p]^{\pm 1}$, and $[(1-p)/p]^{\pm n}$ is nonzero for finite n. Similarly, in the percolation of Example 2, each alteration at a single site introduces the (random) factor given in Eq. (4) (or its inverse) which is related to the Ising model energy; the validity of the next proposition for Example 2 is thus based on the fact that only a finite energy change is induced by spin flips in a finite volume.

For a region $V \subset \mathbb{Z}^d$ and a measurable transformation $\Phi : \Omega \to \Omega$, we say that Φ is V-local if $(\Phi(\omega))_k = \omega_k$ for all $\omega \in \Omega$ and all $k \in V^c$; we say Φ is local if Φ is V-local for some finite V. Note that the value of $(\Phi(\omega))_k$ for $k \in V$ may depend on ω_j's for all $j \in \mathbb{Z}^d$ (including $j \in V^c$).

Proposition 9. If Φ is local and W is any event, then

$$P(W) > 0 \qquad \text{implies} \quad P(\Phi(W)) > 0 \tag{31}$$

Proof. Let V be a finite set such that Φ is V-local and let $S = \{0,1\}^V$. For each $s \in S$, define $\Psi_s : \Omega \to \Omega$ by

$$(\Psi_s \omega)_k = \begin{cases} \omega_k, & k \in V^c \\ s_k, & k \in V \end{cases} \tag{32}$$

Then defining for each $t, s \in S$, $W_{ts} = W \cap \Psi_t(\Omega) \cap \Phi^{-1}(\Psi_s(\Omega))$ so that

$$W = \bigcup_{t,s\in S} W_{ts}, \qquad \Phi(W) = \bigcup_{t,s\in S} \Phi(W_{ts}) = \bigcup_{t,s\in S} \Psi_s(W_{ts})$$

we note that for some $t, s, P(W_{ts}) > 0$ and it then suffices to show that $P(\Psi_s(W_{ts})) > 0$. For $j \in \mathbb{Z}^d$ and $\epsilon = 0$ or 1, define $\Psi_{j,\epsilon}$ by

$$(\Psi_{j,\epsilon} \omega)_k = \begin{cases} \omega_k, & k \neq j \\ \epsilon, & k = j \end{cases} \tag{33}$$

then

$$\Psi_s(W_{ts}) = \Big(\prod_{j\in V}\Psi_{j,s_j}\Big)(W_{ts}) = \prod_{j\in V : s_j\neq t_j}\Psi_{j,s_j}(W_{ts})$$

and (by induction) it suffices to show that for each j and any $U \subset \Omega_j \equiv \{0, 1\}^{\mathbb{Z}^d\setminus\{j\}}$,

$$P(U\times\{\omega_j = t_j\}) > 0 \quad \text{implies} \quad P(U\times\{\omega_j = s_j\}) > 0 \tag{34}$$

but this is the content of Hypothesis C [see Eq. (2)]. ■

Proof of Theorem 1. The random variable N_0 and the events $\{N_0 = k\}$ are invariant under any shift T_j; therefore by Hypothesis B there is some $k_0 = 0, 1, 2, \ldots,$ or ∞ such that $P(N_0 = k_0) = 1$. Suppose $k_0 \neq 0$, 1, or ∞; we wish to obtain a contradiction. Let W_n be the event that $N_0 = k_0$ and each one of the k_0 infinite clusters has nonzero intersection with the cube R_n; then $P(W_n)\to P(N_0 = k_0) = 1$ and so for some m, $P(W_m) > 0$. Let $V = R_m$ and define $\Psi_{\bar{s}}$ as in (32) with $\bar{s}_j = 1\ \forall j\in R_m$; $\Psi_{\bar{s}}$ is the transformation which makes all sites in R_m occupied and leaves all other sites unchanged. By Proposition 9, $P(\Psi_{\bar{s}}(W_m)) > 0$, but clearly $\Psi_{\bar{s}}(W_m) \subset \{N_0 = 1\}$, thus $P(N_0 = 1) > 0$, which contradicts the supposition that $k_0 \neq 0$, 1 or ∞. To prove the second half of the theorem, we first note that if $\rho \equiv P(|C(k)| = \infty) \equiv P(k\in F_0) = 0$; then

$$P(\mathbb{Z}^d\cap F_0 \neq \varnothing) = P\Big(\bigcup_{j\in\mathbb{Z}^d}\{j\in F_0\}\Big) \leqslant \sum_{j\in\mathbb{Z}^d} P(j\in F_0) = 0$$

and so $P(F_0 = \varnothing) = 1 = P(N_0 = 0)$. Finally defining μ_0^j as in (16) we have by Proposition 8 that

$$D(F_0) \equiv \lim_{n\to\infty} n^{-d}\sum_{j\in R_n} T_j\mu_0^0 = E(\mu_0^0) = P(0\in F_0) = \rho \qquad \text{(w.p.o.)} \tag{35}$$

which completes the proof. ■

Proof of Theorem 2. We assume throughout that $N_0 = \infty$ (w.p.o.). Let $\overline{H}_1 = \{C\in H_0 : \underline{D}(C) > 0\}$ and $\overline{N}_1 = |\overline{H}_1|$ and suppose that $P(\overline{N}_1 = 0) \neq 1$; then by ergodicity, as in the proof of Theorem 1, there is some $\bar{k} = 1, 2, \ldots,$ or ∞ such that $P(\overline{N}_1 = \bar{k}) = 1$. We wish to rule out the possibility that $\bar{k} > 1$. Let $\underline{Y} = \sup\{\underline{D}(C) : C\in H_0\}$; then again by ergodicity $\exists D_1 > 0$ so that $P(\underline{Y} = D_1) = 1$ and if $\bar{k} > 1$, then the event

$$U = \{\exists C_1, C_2\in H_0 \text{ with } C_1\neq C_2 \text{ so that } \underline{D}(C_1) + \underline{D}(C_2) > D_1\}$$

has probability 1. Let U_n be the event

$$U_n = \{\exists C_1, C_2\in H_0 \text{ with } C_1\neq C_2 \text{ and } C_1\cap R_n\neq\varnothing, C_2\cap R_n\neq\varnothing, \\ \text{so that } \underline{D}(C_1) + \underline{D}(C_2) > D_1\}$$

then $P(U_n) \to P(U)$ and so for some m, $P(U_m) > 0$. As in the proof of Theorem 1, we consider $\Psi_{\bar{s}}$ defined by (32) with $V = R_m$ and $\bar{s}_j = 1$ $\forall j \in R_m$, and conclude by Proposition 9 that $P(\Psi_{\bar{s}}(U_m)) > 0$, but each $\omega \in \Psi_{\bar{s}}(U_m)$ has an infinite cluster containing $C_1 \cup C_2 \cup R_m$ which has lower density greater than D_1; thus $P(\underline{Y} > D_1) > 0$, which contradicts $P(\underline{Y} = D_1) = 1$ and proves that $\bar{k} = 1$. Letting $\bar{F}_1 = \{j \in \mathbb{Z}^d : j \in C$ for some $C \in \bar{H}_1\}$, we have as in the proof of the second part of Theorem 1 that $\exists \bar{\rho}_1 > 0$ such that $D(\bar{F}_1) = \bar{\rho}_1$ (w.p.o.), but since $\bar{N}_1 = 1$ (w.p.o.) this means that the single $C \in \bar{H}_1$ itself has $D(C) = \bar{\rho}_1$ and thus $\bar{H}_1 = H_1$, $\bar{\rho}_1 = \rho_1$, etc. We are now in the situation where a single cluster has positive density, ρ_1, and all other clusters have zero lower density. To rule out the possibility that any other cluster has positive upper density, we note that otherwise, for some m, the event

$$\tilde{U}_m = \{\exists C_1 \neq C_2 \text{ with } D(C_1) = \rho_1, \bar{D}(C_2) > 0, \\ C_1 \cap R_m \neq \varnothing, C_2 \cap R_m \neq \varnothing\}$$

would have positive probability. But then, as before, $P(\Psi_{\bar{s}}(\tilde{U}_m)) > 0$, and $\Psi_{\bar{s}}(\tilde{U}_m) \subset \{\exists C \in H_0 : \underline{D}(C) > 0, \bar{D}(C) > \rho_1\} \subset \{\bar{D}(\bar{F}_1) > \rho_1\}$, which contradicts the fact that $D(\bar{F}_1) = \rho_1$. We have thus proven that if (with positive probability) there is any infinite cluster with positive lower density, we must actually be in case (ii) of Theorem 2 (w.p.o.).

In order to prove that cases (i), (iii), (iv) exhaust the alternatives, we assume that $\bar{H}_1 = \varnothing$ (w.p.o.) and show that for $l = 2, 3$ either $N_l = 0$ (w.p.o.) or $N_l = \infty$ (w.p.o.). This is not difficult since the proof of Theorem 1 already shows that we need only rule out the case $N_l = 1$ (w.p.o.); but if $N_l = 1$ (w.p.o.) then as in the proof above concerning $\bar{N}_1$ and $\bar{F}_1$; it would follow that $D(F_l) = \rho_l > 0$ and thus the single infinite cluster C in H_l would actually have density $D(C) = \rho_l > 0$, which contradicts the definition of H_l for $l = 2, 3$. This completes the proof of the first part of the theorem; the last part is then proved exactly as was the last part of Theorem 1. ■

Proof of Theorem 3. We define $\Psi_{j,\epsilon}$ as in (33) and note that $\Psi_{0,1}\{0 \in A_{lm}\} = \{0 \in B_{lm}\}$ and $\Psi_{0,0}\{0 \in B_{lm}\} = \{0 \in A_{lm}\}$. Since $P(A_{lm} \neq \varnothing) > 0$ if and only if $\alpha_{lm} \equiv P\{0 \in A_{lm}\} > 0$ and similarly for B_{lm} and β_{lm}, and since clearly $P(A_{lm} \neq \varnothing) = 0$ if (11) is not valid, we see first by Proposition 9 that also $P(B_{lm} \neq \varnothing) = 0$ if (11) is not valid, and second by Proposition 8 that it suffices to assume (11) and conclude that $P(A_{lm} \neq \varnothing) > 0$ in order to complete the proof of the entire theorem; this we proceed to do.

Assuming (11) we have, for some n, that the event

$$W_n \equiv \{\exists C_1 \in H_l, C_2 \in H_m \text{ with } C_1 \neq C_2 \text{ and} \\ C_1 \cap R_n \neq \varnothing, C_2 \cap R_n \neq \varnothing\} \tag{36}$$

has positive probability. We will define an R_n-local transformation Φ such that $\Phi(W_n) \subset \{\exists j \in A_{lm} \cap R_n\} \subset \{A_{lm} \neq \emptyset\}$, which will imply by Proposition 9 that $P\{A_{lm} \neq \emptyset\} > 0$ as desired. We first choose for each $i, j \in \bar{\partial} R_n \equiv \{j \in R_n : \gamma(j, R_n^c) = 1\}$ a subset $R_n^{ij} \subset R_n$ such that

(a) $R_n^{ij} \backslash \{i, j\} \subset R_n \backslash \bar{\partial} R_n$
(b) $\exists$ a nearest-neighbor path in R_n^{ij} which contains and connects i and j
(c) $\exists k \in R_n^{ij}$ so that (b) is false if R_n^{ij} is replaced by $R_n^{ij} \backslash \{k\}$.

Note that if $i = j$ we may take $R_n^{ij} = \{i\}$ and if $\|i - j\| = 1$ we may take $R_n^{ij} = \{i, j\}$. We next choose an ordering $\alpha : \bar{\partial} R_n \to \{1, \ldots, |\bar{\partial} R_n|\}$. We then define $\Phi_1 = \Psi_s$ as in (32) with $V = R_n$ and $s_j = 0\ \forall j \in R_n$ and we define

$$\Phi(\omega) = \begin{cases} \omega & \text{if} \quad \omega \notin W_n \\ \Phi_2(\Phi_1(\omega)) & \text{if} \quad \omega \in W_n \end{cases}$$

where Φ_2 is defined for $\omega \in \Phi_1(W_n)$ as follows: Let $(i, j) = (i(\omega), j(\omega))$ be the $i, j \in \bar{\partial} R_n$ with the smallest value (lexicographically) of the ordering (α_i, α_j) such that $\exists C_1 \in H_l, C_2 \in H_m$ with $C_1 \neq C_2$ such that $\gamma(i, C_1) = 1$, $\gamma(j, C_2) = 1$; such an (i, j) exists by the definitions of W_n and Φ_1. We then define $\Phi_2 = \Psi_s$ analogously to (32) but with $V = R_n$ and

$$s_l = s_l(\omega) = \begin{cases} 1 & \text{if} \quad l \in R_n^{i(\omega), j(\omega)} \\ 0 & \text{otherwise} \end{cases}$$

For $\omega \in \Phi_1(W_n)$, $\Phi_2(\omega)$ has a cutting point which is just the k of part (c) of the definition of R_n^{ij} given above. Thus $\Phi_2(\Phi_1(W_n)) \subset \{A_{lm} \cap R_n \neq \emptyset\}$ as desired and since Φ is R_n-local, the proof is complete. ■

Proof of Theorem 4. Theorem 4 follows from Proposition 8 as in the proof of the last part of Theorem 1. ■

Proof of Theorem 5. Using Proposition 8 as in the proof of the last part of Theorem 1, we see that it suffices in the first part of the theorem to show

$$P(X_0 = 0)/P(X_0 = 1) = P(0 \in F^*)/P(0 \in F)$$

or equivalently

$$P(0 \in F) = P(0 \in (F \cup F^*)) \cdot P(X_0 = 1) \tag{37}$$

and in the second part of the theorem to show that

$$P(0 \in F) \geqslant P(0 \in (F \cup F^*)) \cdot P(X_0 = 1) \tag{38}$$

Now, by Theorem 2, we may replace $(F_1, \partial F_1)$ in (37) and (38) by $(\bar{F}_1, \partial \bar{F}_1)$ without affecting probabilities, where $\bar{F}_1 = \{i : i \in C$ for some $C \in H_0$ with

$\underline{D}(C) > 0\}$. If we define for $\omega \in \Omega$, $\omega[j]$ and $\omega\langle j \rangle$ by

$$(\omega[j])_k = \begin{cases} \omega_k, & k \neq j \\ 0, & k = j \end{cases} \qquad (\omega\langle j \rangle)_k = \begin{cases} \omega_k, & k \neq j \\ 1, & k = j \end{cases}$$

then the following duality is satisfied:

$$\omega \in \{0 \in F\} \Rightarrow \omega[0] \in \{0 \in F^*\}, \qquad \omega \in \{0 \in F^*\} \Rightarrow \omega\langle 0 \rangle \in \{0 \in F\}$$

It then follows that

$$\begin{aligned} \{0 \in F\} &= \{0 \in F \cup F^*\} \cap \{X_0 = 1\}, \\ \{0 \in F^*\} &= \{0 \in F \cup F^*\} \cap \{X_0 = 0\} \end{aligned} \tag{39}$$

and also that for some

$$U \subset \Omega_0 = \{0, 1\}^{\mathbb{Z}^d \setminus \{0\}}, \qquad \{0 \in F \cup F^*\} = U \times \{\omega_0 = 0 \text{ or } 1\} \tag{40}$$

From (40) and the definition of independent percolation [see Eq. (1)], we see that in the first part of the theorem $\{0 \in F \cup F^*\}$ and $\{X_0 = 1\}$ are independent events and thus (37) follows as desired. In the second part of the theorem; we note that for $(F, F^*) = (F_0, \partial F_0)$ or $(\bar{F}_1, \partial \bar{F}_1)$ the random variable

$$Y = \begin{cases} 1 & \text{if} \quad 0 \in F \cup F^* \\ 0 & \text{if} \quad 0 \notin F \cup F^* \end{cases}$$

is an increasing function of the basic occupation variables, $\{X_j\}$, and thus if the FKG inequalities are valid,

$$\operatorname{Cov}(Y, X_0) \equiv E(YX_0) - E(Y)E(X_0) \geqslant 0 \tag{41}$$

But $E(X_0) = P(X_0 = 1)$, $E(Y) = P(0 \in F \cup F^*)$, and by (39), $E(YX_0) = P(0 \in F)$, which yields (38) and completes the proof. ■

Proof of Theorem 6. Define $\bar{\mu}_0^i = \mu_0^i$ [as in (16)] and

$$\bar{\mu}_1^i = \begin{cases} 1 & \text{if} \quad i \in C \text{ for some } C \in H_0 \text{ with } \underline{D}(C) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

Now by Theorem 2, $\bar{\mu}_l^i = \mu_l^i$ (w.p.o.) and so we have from (14)

$$Z_{l,n}^i = \sum_{j \in R_n^i} \left[\bar{\mu}_l^j - E(\bar{\mu}_l^j) \right] \qquad \text{(w.p.o.)} \tag{43}$$

For $l = 0, 1$ the $\bar{\mu}_l^i$'s (but not the μ_1^i's) are increasing functions of the basic occupation variables, $\{X_k\}$, and thus they also satisfy the FKG inequalities [which immediately implies $k_l(i) \geqslant 0$]. Moreover $\{\bar{\mu}_l^i : i \in \mathbb{Z}^d\}$ is translation invariant, and by (18) it is assumed that

$$V_l \equiv \sum_{i \in \mathbb{Z}^d} \operatorname{Cov}(\bar{\mu}_l^0, \bar{\mu}_l^i) < \infty \tag{44}$$

The theorem now follows from the general results of Ref. 19. ∎

The following standard proposition will be used in the proof of Theorem 7. We write W_n i.o. to denote the event that W_n occurs for infinitely many n and we write W_n a.f.o. to denote that W_n occurs for all but finitely many n.

Proposition 10. Let W_n be a sequence of events and Y_n a sequence of random variables; then

$$\begin{aligned} &\sum_{n=1}^{\infty} P(W_n) < \infty \qquad \text{implies} \quad P(W_n \text{ i.o.}) = 0 \\ &\sum_{n=1}^{\infty} E(|Y_n|) < \infty \qquad \text{implies} \quad Y_n \to 0 \qquad \text{(w.p.o.)} \end{aligned} \tag{45}$$

and

$$\begin{aligned} &\liminf_{n\to\infty} P(W_n) = 0 \qquad \text{implies} \quad P(W_n \text{ a.f.o.}) = 0 \\ &\liminf_{n\to\infty} E(|Y_n|) = 0 \qquad \text{implies} \quad \liminf_{n\to\infty} |Y_n| = 0 \qquad \text{(w.p.o.)} \end{aligned} \tag{46}$$

Proof. Equation (45) for W_n is just the Borel–Cantelli lemma. Now by Tchebyshev's inequality, for any $\epsilon > 0$, $\sum P(|Y_n| \geqslant \epsilon) \leqslant \epsilon^{-1} \sum E(|Y_n|)$, and thus letting $W_n = \{|Y_n| \leqslant \epsilon\}$, we have $P(|Y_n| \geqslant \epsilon \text{ i.o.}) = 0$. Thus for each m, $P(\limsup |Y_n| \leqslant 1/m) = 1$, and letting $m \to \infty$ we have $P(\lim |Y_n| = 0) = 1$. To obtain (46) from (45) simply choose an appropriate subsequence. ∎

Proof of Theorem 7. Since

$$P(0 \quad \text{is connected to} \quad \partial R_n) \leqslant \sum_{j \in \partial R_n} G(j) \tag{47}$$

we have from (45) that if (22) is valid then $P(0 \text{ connected to } \partial R_n \text{ a.f.o.}) = 0$ and thus $\rho_0 \equiv P(|C(0)| = \infty) = P(0 \text{ connected to } \partial R_n \ \forall n) = 0$, so by Theorem 1 $H_0 = \varnothing$ (w.p.o.). Now let

$$Y_n = |C(0) \cap R_n| / n^d = n^{-d} \sum_{j \in R_n} \lambda_j \tag{48}$$

where λ_j is the indicator function of the event, $\{0 \text{ is connected to } j\}$, so that

$E(\lambda_j) = G(j)$ and $E(Y_n) = E(|Y_n|) = Q_n$. If (23) is valid, then (for $d > 1$),

$$\sum_n E(|Y_n|) = \sum_n Q_n = \sum_n n^{-d} \sum_{j \in R_n} G(j) = \sum_{k \in \mathbb{Z}^d} \left(G(k) \sum_{n\,:\,R_n \ni k} n^{-d} \right)$$
$$\leqslant \sum_{k \in \mathbb{Z}^d} O(\|k\|^{-(d-1)}) G(k) < \infty \tag{49}$$

and so by (45) $D(C(0)) \equiv \lim Y_n = 0$ (w.p.o.) which implies $\rho_1 = \rho_3 = 0$ and so $\rho_0 = \rho_2$ and $H_0 = H_2$ (w.p.o.). Similarly, if (24) is valid, then $\lim\inf E(|Y_n|) = 0$ and so by (41) $\lim\inf |Y_n| = \underline{D}(C(0)) = 0$ (w.p.o.), so $\rho_1 = 0$ and $H_1 = \emptyset$ (w.p.o.). On the other hand, if $H_0 = H_2$ (w.p.o.), then $\lim Y_n = 0$ (w.p.o.) and since $0 \leqslant Y_n \leqslant 1$, we have by dominated convergence that $Q_n = E(Y_n) \to 0$ as desired. If $H_1 \neq \emptyset$, then we are in case (ii) of Theorem 1 or 2 and so $D(C(0)) \equiv \lim Y_n = 0 \cdot P(0 \in H_2) + \rho_1 \cdot P(0 \in H_1) = (\rho_1)^2$, so by dominated convergence $Q_n = E(Y_n) \to (\rho_1)^2$ as desired. This completes the proof. ■

ACKNOWLEDGMENTS

The authors wish to thank E. Shamir and T. Klein for several useful discussions and for permission to report their results on surface-to-volume ratio prior to publication.

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